

The tangent approximation

2. The tangent plane.

For a function of one variable, $w = f(x)$, the tangent line to its graph at a point (x_0, w_0) is the line passing through (x_0, w_0) and having slope $\left(\frac{dw}{dx}\right)_0$.

For a function of two variables, $w = f(x, y)$, the natural analogue is the *tangent plane* to the graph, at a point (x_0, y_0, w_0) . What's the equation of this tangent plane? Referring to the picture of the graph on the preceding page, we see that the tangent plane

(i) must pass through (x_0, y_0, w_0) , where $w_0 = f(x_0, y_0)$;

(ii) must contain the tangent lines to the graphs of the two partial functions — this will hold if the plane has the same slopes in the **i** and **j** directions as the surface does.

Using these two conditions, it is easy to find the equation of the tangent plane. The general equation of a plane through (x_0, y_0, w_0) is

$$A(x - x_0) + B(y - y_0) + C(w - w_0) = 0.$$

Assume the plane is not vertical; then $C \neq 0$, so we can divide through by C and solve for $w - w_0$, getting

$$(3) \quad w - w_0 = a(x - x_0) + b(y - y_0), \quad a = A/C, \quad b = B/C.$$

The plane passes through (x_0, y_0, w_0) ; what values of the coefficients a and b will make it also tangent to the graph there? We have

$$\begin{aligned} a &= \text{slope of plane (3) in the } \mathbf{i}\text{-direction} && (\text{by putting } y = y_0 \text{ in (3)}); \\ &= \text{slope of graph in the } \mathbf{i}\text{-direction}, && (\text{by (ii) above}) \\ &= \left(\frac{\partial w}{\partial x}\right)_0; && (\text{by the definition of partial derivative}); \quad \text{similarly,} \\ b &= \left(\frac{\partial w}{\partial y}\right)_0. \end{aligned}$$

Therefore the equation of the **tangent plane** to $w = f(x, y)$ at (x_0, y_0) is

$$(4) \quad w - w_0 = \left(\frac{\partial w}{\partial x}\right)_0 (x - x_0) + \left(\frac{\partial w}{\partial y}\right)_0 (y - y_0)$$

3. The approximation formula.

The most important use for the tangent plane is to give an approximation that is the basic formula in the study of functions of several variables — almost everything follows in one way or another from it.

The intuitive idea is that if we stay near (x_0, y_0, w_0) , the graph of the tangent plane (4) will be a good approximation to the graph of the function $w = f(x, y)$. Therefore if the point (x, y) is close to (x_0, y_0) ,

$$(5) \quad \begin{array}{lll} f(x, y) & \approx & w_0 + \left(\frac{\partial w}{\partial x}\right)_0 (x - x_0) + \left(\frac{\partial w}{\partial y}\right)_0 (y - y_0) \\ \text{height of graph} & \approx & \text{height of tangent plane} \end{array}$$

The function on the right side of (5) whose graph is the tangent plane is often called the **linearization** of $f(x, y)$ at (x_0, y_0) : it is the linear function which gives the best approximation to $f(x, y)$ for values of (x, y) close to (x_0, y_0) .

An equivalent form of the approximation (5) is obtained by using Δ notation; if we put

$$\Delta x = x - x_0, \quad \Delta y = y - y_0, \quad \Delta w = w - w_0,$$

then (5) becomes

$$(6) \quad \Delta w \approx \left(\frac{\partial w}{\partial x} \right)_0 \Delta x + \left(\frac{\partial w}{\partial y} \right)_0 \Delta y, \quad \text{if } \Delta x \approx 0, \Delta y \approx 0.$$

This formula gives the approximate change in w when we make a small change in x and y . We will use it often.

The analogous approximation formula for a function $w = f(x, y, z)$ of three variables would be

$$(7) \quad \Delta w \approx \left(\frac{\partial w}{\partial x} \right)_0 \Delta x + \left(\frac{\partial w}{\partial y} \right)_0 \Delta y + \left(\frac{\partial w}{\partial z} \right)_0 \Delta z, \quad \text{if } \Delta x, \Delta y, \Delta z \approx 0.$$

Unfortunately, for functions of three or more variables, we can't use a geometric argument for the approximation formula (7); for this reason, it's best to recast the argument for (6) in a form which doesn't use tangent planes and geometry, and therefore can be generalized to several variables. This is done at the end of this Chapter TA; for now let's just assume the truth of (7) and its higher-dimensional analogues.

Here are two typical examples of the use of the approximation formula. Other examples are in the Exercises. In the rest of your study of partial differentiation, you will see how the approximation formula is used to derive the important theorems and formulas.

Example 1. Give a reasonable square, centered at $(1, 1)$, over which the value of $w = x^3 y^4$ will not vary by more than $\pm .1$.

Solution. We use (6). We calculate for the two partial derivatives

$$w_x = 3x^2 y^4 \quad w_y = 4x^3 y^3$$

and therefore, evaluating the partials at $(1, 1)$ and using (6), we get

$$\Delta w \approx 3\Delta x + 4\Delta y.$$

Thus if $|\Delta x| \leq .01$ and $|\Delta y| \leq .01$, we should have

$$|\Delta w| \leq 3|\Delta x| + 4|\Delta y| \leq .07,$$

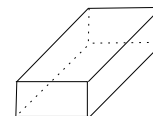
which is within the bounds. So the answer is the square with center at $(1, 1)$ given by

$$|x - 1| \leq .01, \quad |y - 1| \leq .01.$$

Example 2. The sides a, b, c of a rectangular box have lengths measured to be respectively 1, 2, and 3. To which of these measurements is the volume V most sensitive?

Solution. $V = abc$, and therefore by the approximation formula (7),

$$\begin{aligned}\Delta V &\approx bc \Delta a + ac \Delta b + ab \Delta c \\ &\approx 6 \Delta a + 3 \Delta b + 2 \Delta c, \quad \text{at } (1, 2, 3); \end{aligned}$$



thus it is most sensitive to small changes in side a , since Δa occurs with the largest coefficient. (That is, if one at a time the measurement of each side were changed by say .01, it is the change in a which would produce the biggest change in V , namely .06 .)

The result may seem paradoxical — the value of V is most sensitive to the length of the *shortest* side — but it's actually intuitive, as you can see by thinking about how the box looks.

Sensitivity Principle *The numerical value of $w = f(x, y, \dots)$, calculated at some point (x_0, y_0, \dots) , will be most sensitive to small changes in that variable for which the corresponding partial derivative w_x, w_y, \dots has the largest absolute value at the point.*

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